

Effective-Mass Klein-Gordon-Yukawa Problem for Bound and Scattering States

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Abstract

Bound and scattering state solutions of the effective-mass Klein-Gordon equation are obtained for the Yukawa potential with any angular momentum ℓ . Energy eigenvalues, normalized wave functions and scattering phase shifts are calculated as well as for the constant mass case. Bound state solutions of the Coulomb potential are also studied as a limiting case. Analytical and numerical results are compared with the ones obtained before.

Keywords: Yukawa potential, Coulomb potential, Klein-Gordon equation, Position-Dependent Mass, Bound State, Scattering State

PACS numbers: 03.65.N, 03.65.Ge, 03.65.Nk, 03.65.Pm, 03.65.-w, 12.39.Fd

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I. INTRODUCTION

In the view of relativistic quantum mechanics, a particle moving in a potential field is described particularly with the Klein-Gordon (KG) equation. Solutions of the one-dimensional KG equation have been received great attention for some potentials [1-2]. The relativistic quantum mechanical problems that can be solved exactly are very restricted [3]. In the present work, we obtain approximate analytical energy eigenvalues, normalized wave functions and scattering phase shifts for the Yukawa potential [4]

$$V(r) = -\frac{\eta}{r} e^{-\alpha r}. \quad (1)$$

where α is the screening parameter and η is the strength of the potential. The Yukawa potential has many applications in different areas of physics: in high-energy physics as a short-range potential [4], atomic and molecular physics as a screened Coulomb potential and plasma physics as the Debye-Hückel potential [5]. In recent years, considerable efforts have also been made to study the approximate bound state solutions of the Yukawa potential in the non-relativistic domain by using different methods [5-9].

On the other hand, the position-dependent mass (PDM) formalism [10] has many applications in different areas, such as impurities in crystals [11], the study of quantum wells and quantum dots [12] and semiconductor heterostructures [13]. In recent years, the relativistic extension of the position-dependent mass formalism has been studied by many authors for different types of potentials [14-16].

The organization of this work is as follows. In Section II, we study the approximate bound state solutions and corresponding normalized wave functions for the Yukawa potential. We list some numerical results for the cases of PDM and constant mass presented in Table I and II. In Section III, we deal with the approximate scattering state solutions of the Yukawa potential and give analytical expressions for the phase shifts. In Section IV, we give our conclusions.

II. BOUND STATE SOLUTIONS

The radial part of the effective-mass KG equation is written as [17]

$$\frac{d^2\phi(r)}{dr^2} - \left\{ \frac{\ell(\ell+1)}{r^2} + \frac{1}{\hbar^2 c^2} [m^2(r)c^4 - (E^2 - 2EV(r) + V^2(r))] \right\} \phi(r) = 0, \quad (2)$$

where ℓ is the angular momentum quantum number, E is the energy of the particle and c is the velocity of the light.

In recent years, the following approximation

$$\frac{1}{r^2} \approx 4\alpha^2 \frac{e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2}, \quad (3)$$

has been used [18, 19] instead of the centrifugal term in the wave equations to obtain the solutions with any ℓ values. It has a good accuracy for small values of the potential parameter α [18, 19]. A remarkable approximation is proposed by Alhaidari [20] where the author suggested, for the first time, an approximation for the orbital term $1/r$ in the Dirac equation not for the $1/r^2$ term. By using this approximation, it is possible to find the approximate bound state solutions of the Dirac equation for coupling to pure $1/r$ vector potentials with any ℓ values for higher excitation levels with more accuracy than using the traditional approximation for the $1/r^2$ term [20].

We define the mass function as

$$m(r) = m_0 + \frac{m_1}{e^{2\alpha r} - 1}, \quad (4)$$

where m_0 and m_1 are two parameters and m_0 will correspond to the rest mass of the KG particle. Using the approximation given in Eq. (3) Yukawa potential becomes

$$V(r) = -2\alpha\eta \frac{e^{-2\alpha r}}{1 - e^{-2\alpha r}}, \quad (5)$$

Inserting Eqs. (4) and (5) into Eq. (2) and taking a new variable $z = (1 - e^{-2\alpha r})^{-1}$ ($z \rightarrow \infty$ for $r \rightarrow 0$ and $z \rightarrow 1$ for $r \rightarrow \infty$), we obtain

$$\begin{aligned} & z(1-z) \frac{d^2\phi(z)}{dz^2} + (1-2z) \frac{d\phi(z)}{dz} \\ & + \left\{ \ell(\ell+1) - \frac{\beta^2}{4\alpha^2} (m_0^2 c^4 - E^2) \frac{1}{z(1-z)} + \left(\frac{\beta^2 m_0 m_1 c^4}{2\alpha^2} - \frac{E\beta^2 \eta}{2\alpha} \right) \frac{1}{z} \right. \\ & \left. - \left(\frac{\beta^2 m_1^2 c^4}{4\alpha^2} - \beta^2 \eta^2 \right) \frac{1-z}{z} \right\} \phi(z) = 0, \end{aligned} \quad (6)$$

where $\beta^2 = 1/\hbar^2 c^2$. Taking the form of the wave function

$$\phi(z) = z^{\lambda_1} (1-z)^{\lambda_2} \psi(z), \quad (7)$$

and inserting into Eq. (6), one gets a hypergeometric-type equation [21]

$$\begin{aligned} & z(1-z) \frac{d^2\psi(z)}{dz^2} + [1 + 2\lambda_1 - 2(\lambda_1 + \lambda_2 + 1)z] \frac{d\psi(z)}{dz} \\ & + \left\{ -\lambda_1^2 - \lambda_2^2 - \lambda_1 - \lambda_2 - 2\lambda_1\lambda_2 + \ell(\ell+1) + \frac{\beta^2 m_1^2 c^4}{4\alpha^2} - \beta^2 \eta^2 \right\} \psi(z) = 0, \end{aligned} \quad (8)$$

where

$$\lambda_1^2 = \frac{\beta^2}{4\alpha^2} (m_0^2 c^4 - E^2) - \frac{\beta^2 m_0 m_1 c^4}{2\alpha^2} + \frac{\beta^2 m_1^2 c^4}{4\alpha^2} + \frac{E\beta^2 \eta}{2\alpha} - \beta^2 \eta^2, \quad (9)$$

$$\lambda_2^2 = \frac{\beta^2}{4\alpha^2} (m_0^2 c^4 - E^2). \quad (10)$$

Comparing Eq. (8) with the hypergeometric equation of the following form [21]

$$z(1-z)y'' + [\xi_3 - (\xi_1 + \xi_2 + 1)z]y' - \xi_1\xi_2y = 0, \quad (11)$$

we find the solution of Eq. (8) as the hypergeometric function

$$\psi(z) = {}_2F_1(\xi_1, \xi_2; \xi_3; z). \quad (12)$$

where

$$\xi_1 = \lambda_1 + \lambda_2 + \frac{1}{2} \left(1 + \sqrt{1 + 4\ell(\ell+1) + \frac{\beta^2 m_1^2 c^4}{\alpha^2} - 4\beta^2 \eta^2} \right), \quad (13)$$

$$\xi_2 = \lambda_1 + \lambda_2 + \frac{1}{2} \left(1 - \sqrt{1 + 4\ell(\ell+1) + \frac{\beta^2 m_1^2 c^4}{\alpha^2} - 4\beta^2 \eta^2} \right), \quad (14)$$

$$\xi_3 = 1 + 2\lambda_1. \quad (15)$$

From Eq. (7), we obtain total wave function

$$\phi(z) = Nz^{\lambda_1}(1-z)^{\lambda_2} {}_2F_1(\xi_1, \xi_2, \xi_3, z). \quad (16)$$

where N is normalization constant that will be derived in Appendix A. When either ξ_1 or ξ_2 equals to a negative integer $-n$, the hypergeometric function $\psi(z)$ can be reduced to a finite solution. This gives us a polynomial of degree n in Eq. (12) and the following quantum condition

$$\lambda_1 + \lambda_2 + \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4\ell(\ell+1) + \frac{\beta^2 m_1^2 c^4}{\alpha^2} - 4\beta^2 \eta^2} = -n \quad (n = 0, 1, 2, \dots). \quad (17)$$

It is the relativistic energy eigenvalue equation for the Yukawa potential within the PDM formalism. Defining two new parameters such as

$$\mathcal{L}(\ell) = \sqrt{1 + 4\ell(\ell+1) + \frac{\beta^2 m_1^2 c^4}{\alpha^2} - 4\beta^2 \eta^2}, \quad (18)$$

$$\lambda_1^2 = \frac{\beta^2}{4\alpha^2} (m_0^2 c^4 - E^2) + \frac{E\beta^2 \eta}{2\alpha} + \Lambda(\beta); \quad \Lambda(\beta) = -\frac{\beta^2 m_0 m_1 c^4}{2\alpha^2} + \frac{\beta^2 m_1^2 c^4}{4\alpha^2} - \beta^2 \eta^2, \quad (19)$$

we get the approximate energy eigenvalues as

$$\begin{aligned} E_{n\ell}^{(\mp)} &= \frac{\alpha\eta}{2[(\mathcal{L}(\ell) + 2n + 1)^2 + \beta^2 \eta^2]} \left\{ [\mathcal{L}(\ell) + 2n + 1]^2 - 4\Lambda(\beta) \right. \\ &\mp \frac{\mathcal{L}(\ell) + 2n + 1}{\beta\eta} \\ &\times \left. \sqrt{\frac{4\beta^2 m_0^2}{\alpha^2} [\mathcal{L}(\ell) + 2n + 1]^2 + \beta^2 \eta^2} - [[\mathcal{L}(\ell) + 2n + 1]^2 - 4\Lambda(\beta)]^2 \right\}. \end{aligned} \quad (20)$$

Table I presents the comparison of our numerical results for the case of constant mass ($m_1 = 0$) with the ones given in Ref. [22]. We restrict ourselves for only s -states and take $m_0 = 1$ because of

the computation in Ref. [22]. Our parameters η and α correspond to λ and $k(\equiv \omega\lambda)$, respectively. The relativistic energy is obtained as $E_R = \eta_{exact} \left(\sqrt{1 - \lambda^2} - 1 \right) + 1$ in Ref. [22]. The same numerical values of η_{exact} is used with Ref. [22] to compare our numerical results. We plot the $1/r$ and the approximation $2\alpha e^{-\alpha r}(1 - e^{-\alpha r})$ versus r . It seems that the energy eigenvalues have a good accuracy up to the values of $\eta \leq 0.25$ and $\alpha \leq 0.30$. Table II presents numerical energy eigenvalues for the case of position dependent mass including also for the case of constant mass with different values of (n, ℓ) .

Now, let us study the results of our formalism for the case of the Coulomb potential.

1. Relativistic-Coulomb Limit

In the limiting case $\alpha \rightarrow 0$, the Yukawa potential reduces to

$$V(r) = -\frac{\eta}{r}, \quad (21)$$

which is an attractive Coulomb potential received great interest not only in the case of constant mass [23-25] but also within the position-dependent mass formalism [26].

In order to compare our results for the bound states with the ones obtained in Ref. [27], we expand the mass function, Eq. (4), into Taylor series

$$m(r) \xrightarrow{\alpha \rightarrow 0} m_0 - \frac{m_1}{2} + \frac{m_1}{2\alpha r} + \dots \equiv M_0 + \frac{M_1}{r} + \dots, \quad (22)$$

The parameters b and m_0 used in Ref. [27] are defined as $b \rightarrow M_1, m_0 \rightarrow M_0$. Taking the vector part of the potential which is equal to the scalar part as stated in Ref. [27] as $V(r) = S(r) = -\frac{\eta}{r}$ and inserting the mass function into the KG equation including the scalar potential

$$\frac{d^2\phi(r)}{dr^2} - \left\{ \frac{\ell(\ell+1)}{r^2} + \frac{1}{\hbar^2 c^2} [m^2(r)c^4 + 2m(r)S(r) + S^2(r) - (E^2 - 2EV(r) + V^2(r))] \right\} \phi(r) = 0, \quad (23)$$

gives the following equation

$$\left\{ \frac{d^2}{dr^2} - A_1^2 - A_2 \frac{1}{r} - A_3 \frac{1}{r^2} \right\} \phi(r) = 0, \quad (24)$$

where

$$A_1^2 = \beta^2(M_0^2 c^4 - E^2); A_2 = \beta^2(2M_0 M_1 c^4 - 2M_0 c^2 \eta - 2E\eta); A_3 = \beta^2(M_1^2 c^4 - 2M_1 c^2 \eta + \beta \ell(\ell+1)), \quad (25)$$

The wave function is written as

$$\phi(r) = r^\kappa e^{-A_1 r} f(r). \quad (26)$$

where we set $\kappa(\kappa - 1) = A_3$. Thus, inserting Eq. (26) into Eq. (24) and using a new transformation $z = 2A_1 r$, we obtain

$$z \frac{d^2 f(z)}{dz^2} + (2\kappa - z) \frac{df(z)}{dz} + \left(-\kappa - \frac{A_2}{2A_1}\right) f(z) = 0. \quad (27)$$

which has the form of the Kummer differential equation [21]

$$xy''(x) + (c - x)y'(x) - ay(x) = 0. \quad (28)$$

So, the solution of Eq. (27) is given by

$$f(z) \sim {}_1F_1 \left(\kappa + \frac{A_2}{2A_1}; 2\kappa, z \right). \quad (29)$$

In order to get a finite solution, the following condition must be satisfied

$$\kappa + \frac{A_2}{2A_1} = -n \quad (n = 0, 1, 2, \dots). \quad (30)$$

We get the bound state energy eigenvalues for the Coulomb potential as

$$E_{n\ell}^{Coul.} = \frac{M_0 c^2}{4\beta^2 \eta^2 + \left[N + \sqrt{1 + \eta' + 4\beta^2 \ell(\ell + 1)} \right]^2} \times \left\{ 4\beta^2 \eta (M_1 c^2 - \eta) \right. \\ \left. + \left(N + \sqrt{1 + \eta' + 4\beta^2 \ell(\ell + 1)} \right) \sqrt{\left[N + \sqrt{1 + \eta' + 4\beta^2 \ell(\ell + 1)} \right]^2 - \eta' (M_1 c^2 - 2\eta)} \right\}. \quad (31)$$

where

$$N = 2n + 1 \quad \text{and} \quad \eta' = 4\beta M_1 c^2 (M_1 c^2 - 2\eta). \quad (32)$$

This result is the same with the one for $\ell = 0$ given in Ref. [27].

III. SCATTERING STATE SOLUTIONS

Now we turn to the solution of the Eq. (2) to obtain the scattering states for the Yukawa potential. We use a new variable $s = 1 - e^{-2\alpha r}$ ($s \rightarrow 0$ for $r \rightarrow 0$ and $s \rightarrow 1$ for $r \rightarrow \infty$) and obtain

$$s(1-s) \frac{d^2 \phi(s)}{ds^2} - s \frac{d\phi(s)}{ds} \\ + \left\{ \frac{E\beta^2 \eta}{2\alpha} - \frac{\beta^2}{2\alpha^2} (m_0 m_1 c^4 - \ell(\ell + 1)) \frac{1}{s} - \frac{\beta^2}{4\alpha^2} (m_0^2 c^4 - E^2) \frac{s}{1-s} \right. \\ \left. + \left(\beta^2 \eta^2 - \frac{\beta^2 m_1^2 c^4}{4\alpha^2} \right) \frac{1-s}{s} \right\} \phi(s) = 0. \quad (33)$$

Defining the trial wave function

$$\phi(s) = s^{k_1} (1-s)^{k_2} \psi(s), \quad (34)$$

and substituting into Eq. (33), we obtain a hypergeometric-type equation for $\psi(s)$

$$\begin{aligned} & s(1-s) \frac{d^2 \psi(s)}{ds^2} + [2k_1 - (2k_1 + 2ik'_2 + 1)s] \frac{d\psi(s)}{ds} \\ & + \left\{ -2ik_1 k'_2 - k_1 - \ell(\ell+1) - \frac{\beta^2 m_1^2 c^4}{2\alpha^2} + \frac{E\beta^2 \eta}{2\alpha} \right\} \psi(s) = 0, \end{aligned} \quad (35)$$

where

$$k_1 = \frac{1}{2} \left\{ 1 + \sqrt{1 + 4\ell(\ell+1) + \frac{\beta^2 m_1^2 c^4}{\alpha^2} - 4\beta^2 \eta^2} \right\}, \quad (36)$$

$$k_2 = ik'_2; \quad k'_2 = \sqrt{\frac{\beta^2}{4\alpha^2} (E^2 - m_0^2 c^4)}. \quad (37)$$

The solution of Eq. (35) is a hypergeometric function

$$\psi(s) = {}_2F_1(p, q; r; s), \quad (38)$$

where

$$p = k_1 + ik'_2 + \sqrt{\frac{\beta^2 m_1^2 c^4}{4\alpha^2} - \beta^2 \eta^2 - \frac{\beta^2}{4\alpha^2} (E^2 - m_0^2 c^4) - \frac{\beta^2 m_0 m_1 c^4}{2\alpha^2} + \frac{E\beta^2 \eta}{2\alpha}}, \quad (39)$$

$$q = k_1 + ik'_2 - \sqrt{\frac{\beta^2 m_1^2 c^4}{4\alpha^2} - \beta^2 \eta^2 - \frac{\beta^2}{4\alpha^2} (E^2 - m_0^2 c^4) - \frac{\beta^2 m_0 m_1 c^4}{2\alpha^2} + \frac{E\beta^2 \eta}{2\alpha}}, \quad (40)$$

$$r = 2k_1. \quad (41)$$

From Eqs. (34) and (38), we write the wave function of the scattering states

$$\phi(s) = s^{k_1} (1-s)^{ik'_2} {}_2F_1(p, q; r; s), \quad (42)$$

or

$$\phi(r) = (1 - e^{-2\alpha r})^{k_1} e^{-2ik'_2 \alpha r} {}_2F_1(p, q; r; 1 - e^{-2\alpha r}). \quad (43)$$

To obtain a finite solution, p or q must be a negative integer. This gives the following equality

$$k_1 + ik'_2 + \sqrt{\frac{\beta^2 m_1^2 c^4}{4\alpha^2} - \beta^2 \eta^2 - \frac{\beta^2}{4\alpha^2} (E^2 - m_0^2 c^4) - \frac{\beta^2 m_0 m_1 c^4}{2\alpha^2} + \frac{E\beta^2 \eta}{2\alpha}} = -n, \quad (n = 0, 1, 2, \dots) \quad (44)$$

which is the same energy eigenvalue equation given in Eq. (17). We write the asymptotic form of the wave function given in Eq. (43) to obtain the scattering phase shifts. For this purpose, we use

the property of the hypergeometric functions [21]

$$\begin{aligned} {}_2F_1(a, b; c; y) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b; a+b-c+1; 1-y) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-y)^{c-a-b} {}_2F_1(c-a, c-b; c-a-b+1; 1-y), \end{aligned} \quad (45)$$

and ${}_2F_1(a, b; c; 0) = 1$, we obtain the wave function for the limit of $r \rightarrow \infty$

$$\begin{aligned} \phi(r \rightarrow \infty) &\rightarrow (1 - e^{-2\alpha r})^{k_1} \left\{ \frac{\Gamma(2k_1)\Gamma(-2ik'_2)}{\Gamma(k_1 - ik'_2 - \mathcal{A}(k_1, k_2))\Gamma(k_1 - ik'_2 + \mathcal{A}(k_1, k_2))} e^{-2ik'_2\alpha r} \right. \\ &\quad \left. + \frac{\Gamma(2k_1)\Gamma(2ik'_2)}{\Gamma(k_1 + ik'_2 - \mathcal{A}(k_1, k_2))\Gamma(k_1 + ik'_2 + \mathcal{A}(k_1, k_2))} e^{2ik'_2\alpha r} \right\}, \end{aligned} \quad (46)$$

which could be written

$$\begin{aligned} \phi(r \rightarrow \infty) &\rightarrow (1 - e^{-2\alpha r})^{k_1} \Gamma(2k_1) \left\{ \left[\frac{\Gamma(2ik'_2)}{\Gamma(k_1 + ik'_2 - \mathcal{A}(k_1, k_2))\Gamma(k_1 + ik'_2 + \mathcal{A}(k_1, k_2))} \right]^* e^{-2ik'_2\alpha r} \right. \\ &\quad \left. + \frac{\Gamma(2ik'_2)}{\Gamma(k_1 + ik'_2 - \mathcal{A}(k_1, k_2))\Gamma(k_1 + ik'_2 + \mathcal{A}(k_1, k_2))} e^{2ik'_2\alpha r} \right\}, \end{aligned} \quad (47)$$

where

$$\mathcal{A}(k_1, k_2) = \sqrt{\frac{\beta^2 m_1^2 c^4}{4\alpha^2} - \beta^2 \eta^2 - \frac{\beta^2}{4\alpha^2} (E^2 - m_0^2 c^4) - \frac{\beta^2 m_0 m_1 c^4}{2\alpha^2} + \frac{E\beta^2 \eta}{2\alpha}}. \quad (48)$$

From Eq. (47) we obtain

$$\begin{aligned} \phi(r \rightarrow \infty) &\rightarrow 2(1 - e^{-2\alpha r})^{k_1} \Gamma(2k_1) \left| \frac{\Gamma(2ik'_2)}{\Gamma(k_1 + ik'_2 - \mathcal{A}(k_1, k_2))\Gamma(k_1 + ik'_2 + \mathcal{A}(k_1, k_2))} \right| \\ &\sin \left(2\alpha k'_2 r + \frac{\pi}{2} + \arg \frac{\Gamma(2ik'_2)}{\Gamma(k_1 + ik'_2 - \mathcal{A}(k_1, k_2))\Gamma(k_1 + ik'_2 + \mathcal{A}(k_1, k_2))} \right), \end{aligned} \quad (49)$$

and, consequently, the phase shifts δ_ℓ are obtained as

$$\delta_\ell = (\ell + 1) \frac{\pi}{2} + \delta = (\ell + 1) \frac{\pi}{2} + \arg \frac{\Gamma(2ik'_2)}{\Gamma(k_1 + ik'_2 - \mathcal{A}(k_1, k_2))\Gamma(k_1 + ik'_2 + \mathcal{A}(k_1, k_2))}. \quad (50)$$

IV. CONCLUSION

We have studied the approximate bound and scattering state solutions of the effective mass KG equation for the Yukawa potential. We have obtained the energy eigenvalues, normalized wave functions and scattering phase shifts approximately as well as for the constant mass case. We have presented our numerical results of the energy eigenvalues in Tables I and II. We have compared them for the constant mass case with the ones obtained in the literature. We have also studied the bound state solutions of the Coulomb potential obtained from the limiting case of $\alpha \rightarrow 0$ with the position-dependent and constant mass cases. We have observed that the results obtained for the Coulomb potential are the same with the ones obtained in the literature.

V. ACKNOWLEDGMENTS

This research was partially supported by the Scientific and Technical Research Council of Turkey.

Appendix A: Derivation of Normalization Constant

By using Eq. (17), the wavefunction in Eq. (16) is written

$$\phi(z) = Nz^{\lambda_1}(1-z)^{\lambda_2} {}_2F_1(-n, n+2\lambda_1+2\lambda_2+1; 1+2\lambda_1; z), \quad (\text{A1})$$

We use a new variable $s = z^{-1}$ ($s \rightarrow 0$ for $z \rightarrow \infty$ and $s \rightarrow 1$ for $z \rightarrow 1$) to normalize the wavefunction. For this purpose, we substitute $s \rightarrow \frac{2}{1-s}$

$$\phi\left(\frac{1-s}{2}\right) = N'(1-s)^{\lambda_1}(1+s)^{\lambda_2} {}_2F_1(-n, n+2\lambda_1+2\lambda_2+1; 1+2\lambda_1; \frac{1-s}{2}), \quad (\text{A2})$$

where $N' = N2^{-(\lambda_1+\lambda_2)}(-1)^{2\lambda_2}$.

Using the following equality [21]

$$P_n^{(\alpha, \beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1(-n, n+\alpha+\beta+1; 1+\alpha; \frac{1-x}{2}), \quad (\text{A3})$$

the hypergeometric functions in Eq. (A2) could be written in terms of the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$. Here, $(\kappa)_n$ is defined $(\kappa)_n = \frac{\Gamma(\kappa+n)}{\Gamma(\kappa)}$, and setting $\alpha = 2\lambda_1$ and $\beta = 2\lambda_2$ in Eq. (A3), we rewrite Eq. (A2)

$$\phi\left(\frac{1-s}{2}\right) = N'(1-s)^{\lambda_1}(1+s)^{\lambda_2} n! \frac{\Gamma(1+2\lambda_1)}{\Gamma(n+1+2\lambda_1)} P_n^{(2\lambda_1, 2\lambda_2)}(s). \quad (\text{A4})$$

where the Jacobi polynomials $P_n^{(a, b)}(z)$ are defined [21]

$$P_n^{(a, b)}(x) = \frac{1}{n!} \sum_{k=0}^n \frac{\Gamma(-n+k)}{\Gamma(-n)} \frac{\Gamma(a+b+n+k+1)}{\Gamma(a+b+n+1)} \frac{\Gamma(a+n+1)}{\Gamma(a+k+1)} \frac{1}{k!} (1-x)^k 2^{-k}, \quad (\text{A5})$$

Using Eq. (A5) in Eq. (A4) we obtain the wavefunction

$$\phi\left(\frac{1-s}{2}\right) = N' \Sigma(n, k) (1-s)^{\lambda_1+k} (1+s)^{\lambda_2}, \quad (\text{A6})$$

where

$$\Sigma(n, k) = \Gamma(1+2\lambda_1) \sum_{k=0}^n \frac{1}{2^k k!} \frac{\Gamma(-n+k)}{\Gamma(-n)} \frac{\Gamma(2\lambda_1+2\lambda_2+n+k+1)}{\Gamma(2\lambda_1+2\lambda_2+n+1)} \frac{1}{\Gamma(2\lambda_1+k+1)}. \quad (\text{A7})$$

The normalization condition $\int_0^1 \left| \phi\left(\frac{1-s}{2}\right) \right|^2 ds = 1$ gives

$$|N'|^2 |\Sigma(n, k)|^2 \int_0^1 (1-s)^{2\lambda_1+2k} (1+s)^{2\lambda_2} ds = 1. \quad (\text{A8})$$

Comparing the last integral with the following [21]

$$\int_0^1 t^{\delta-1} (1-t)^{\nu-\delta-1} (1-zt)^{-\gamma} dt = \frac{\Gamma(\delta)\Gamma(\nu-\delta)}{\Gamma(\nu)} {}_2F_1(\gamma, \delta; \nu; z), \quad (\text{A9})$$

and setting $\delta = 1$, $z = -1$, $\gamma = -2\lambda_2$ and $\nu = 2\lambda_1 + 2k + 2$ we find the normalization constant as

$$|N'|^2 = \frac{2\lambda_1 + 2k + 1}{|\Sigma(n, k)|^2 {}_2F_1(-2\lambda_2, 1; 2\lambda_1 + 2k + 2; -1)}. \quad (\text{A10})$$

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TABLE I: Comparison of Klein-Gordon ground state energies for different parameter values.

η	α	η_{exact}	Ref. [22]	our results
0.125	0.01250	0.83072460	0.993484	0.998702
	0.06250	0.30947218	0.997573	0.999999
	0.09375	0.12370738	0.999030	0.999542
	0.12500	0.02452195	0.999808	0.998100
	0.14375	0.00187260	0.999985	0.996759
0.25	0.0250	0.88881431	0.971776	0.994130
	0.1250	0.35655334	0.998678	0.999960
	0.1875	0.15650395	0.995030	0.998556
	0.2500	0.04068600	0.998708	0.993138
	0.3000	0.00257940	0.999918	0.985799

 TABLE II: Energy eigenvalues of the Yukawa potential for different values of n and ℓ (in $\hbar = m_0 = c = 1$ unit).

n	ℓ	η	α	$E_{n\ell}^{(+)}$		$-E_{n\ell}^{(-)}$	
				$m_1 = 0$	$m_1 = 0.1$	$m_1 = 0$	$m_1 = 0.1$
0	0	0.1	0.01	0.999181	0.411464	0.998173	0.394898
		0.01	0.1	0.995475	0.859773	0.994475	0.855513
1	0	0.1	0.01	0.999987	0.614868	0.998985	0.602709
		0.01	0.1	0.980294	0.900967	0.979294	0.898992
	1	0.1	0.01	0.999911	0.638787	0.998910	0.627267
		0.01	0.1	0.954438	0.887484	0.953438	0.885983
2	0	0.1	0.01	0.999913	0.712338	0.998912	0.702947
		0.01	0.1	0.954440	0.884025	0.953440	0.882563
	1	0.1	0.01	0.999622	0.725851	0.998622	0.716879
		0.01	0.1	0.917015	0.852051	0.916015	0.850766
	2	0.1	0.01	0.999200	0.748196	0.998199	0.739937
		0.01	0.1	0.866525	0.801327	0.865525	0.800141
10	0	0.1	0.01	0.994432	0.888765	0.993432	0.885794
	5	0.1	0.01	0.987613	0.897157	0.986613	0.894680
	10	0.1	0.01	0.978199	0.900957	0.977199	0.898987

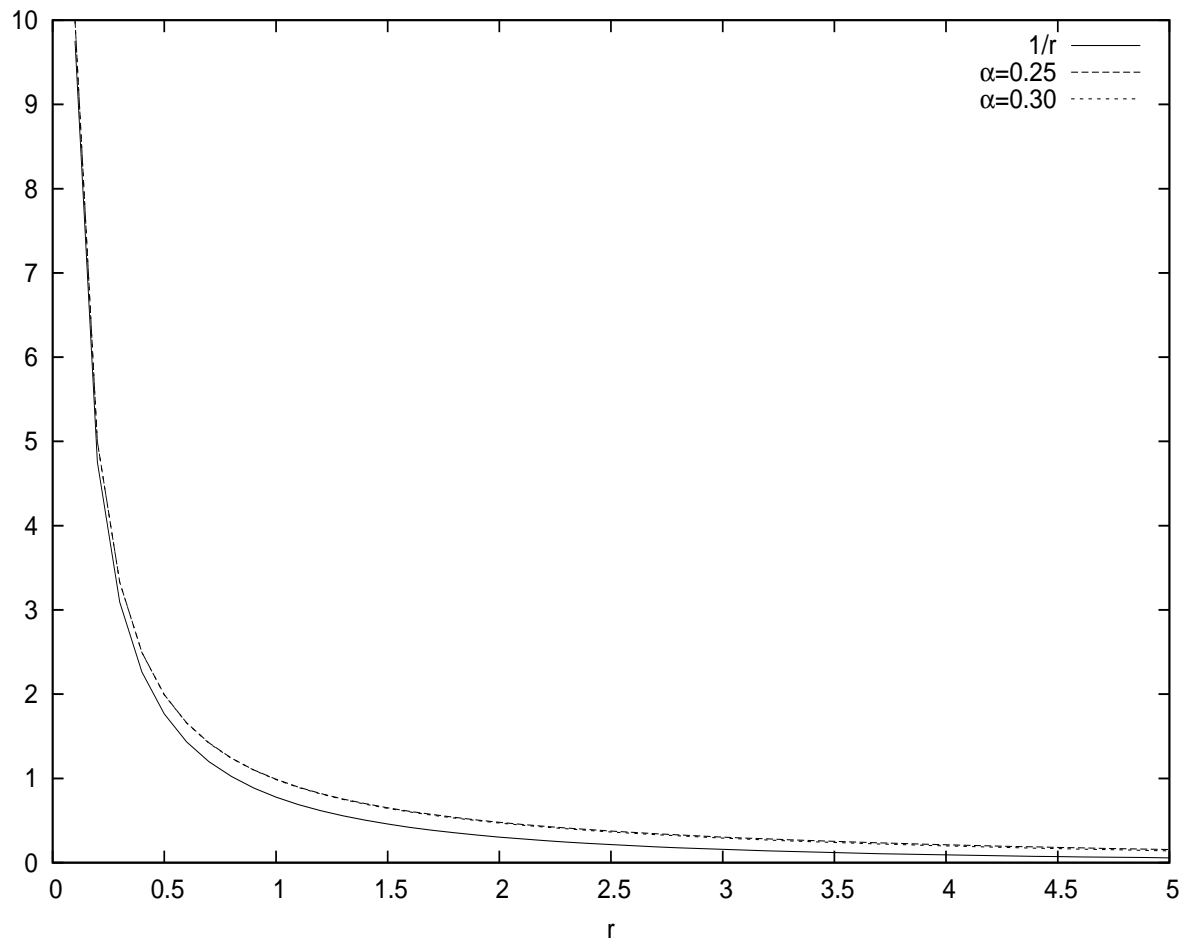


FIG. 1: Comparison of $1/r$ (full line) with $2\alpha e^{-\alpha r}(1 - e^{-\alpha r})$ for different values of $\alpha = 0.25, 0.30$, respectively.